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Wrapped and unwrapped phase of radiation scattered by a discrete number of particles

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Abstract

This paper investigates wrapped and unwrapped phase differences generated by a non-Gaussian scattering model: the two-dimensional random walk. Mean square values for these quantities are obtained for one and two scatterers, as well as the large scatterer limit when the field constitutes a circular complex Gaussian process. Numerical simulation is used to investigate the phase under more general fluctuation conditions, and reveals that the wrapped phase difference correlation converges rapidly to that result predicted for a Gaussian speckle field. Analytical results for the unwrapped phase indicate that this quantity transitions from a stationary process for one and two scatterers to a non-stationary process in the large scatterer limit. The nature of this transition is examined using numerical simulation for arbitrary scatterer number. Phase correlations are of consequence in various phase sensitive detection systems, and this paper examines both Gaussian and non-Gaussian fields.

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1. Introduction

The coherence properties of scattered radiation are of fundamental physical interest and influence the performance of various communication and remote sensing systems [1, 2]. Intensity fluctuations and correlations are of principal interest in direct detection systems, which respond solely to intensity, and a number of results exist for Gaussian as well as non-Gaussian fields [3, 4]. Phase-sensitive or heterodyne detection systems, however, are also sensitive to fluctuations in the field's phase [4, 5]. These are primarily used to detect Doppler shifts in scattered radiation, and thus determine target velocity and vibration. Phase sensitive detection has also been used to investigate refractive index fluctuations in the atmosphere [6]. In comparison to intensity, few results exist for the behaviour of phase beyond the Gaussian limit. The aim of this paper is to extend investigations into non-Gaussian regimes by considering the phase properties of a well-known scattering model, namely the two-dimensional random walk [7], which returns the scattered field as

$$E(\tilde{r}, t) = \sum_{n=1}^{N} a_n(\tilde{r}, t)\exp[\mathrm{i}\theta_n(\tilde{r}, t)]\exp(\mathrm{i}\omega t) = \sqrt{I(\tilde{r}, t)}\exp[\mathrm{i}\phi(\tilde{r}, t)]\exp(\mathrm{i}\omega t), \tag{1}$$

where the a_ns are real scattering amplitudes, the θ_ns are independent scatterer phase shifts and ω is the radiation frequency whose dependence we shall suppress. While the phase shifts θ can vary without bound in direct proportion to the distance between scatterers and detector, the complex exponentials in the above sum effectively confine the phase ϕ of the resultant field to a 2π range, conveniently taken to be $[-\pi, \pi)$; we will refer to this as the field's wrapped phase. However, when a sequence of measurements is made the phase may be unwrapped so that it extends outside the 2π range.

Many phase sensitive instruments operate by comparing field measurements obtained at different times, frequencies or spatial locations, e.g., interferometers typically output the product of field values $E(t_2)E^*(t_1) = \sqrt{I_2 I_1}\exp[\mathrm{i}(\phi_2 - \phi_1)]$ [8, 9]. While the difference $(\phi_2 - \phi_1)$ can vary over $[-2\pi, 2\pi)$, the phase returned from the instrument is the phase of $E(t_2)E^*(t_1)$, which is confined to $[-\pi, \pi)$ and will be referred to as the field's wrapped phase difference $\varphi_{2\pi}(\tau)$. This quantity can also be visualized as the angle between field vectors $E(t_1)$ and $E(t_2)$ in the complex plane. For some applications, however, the phase needs to be unwrapped, e.g. in measurements of refractive index variations when the optical path length changes by more than a wavelength. Consideration of the field rotation between times t_1 and t_2 allows one to define an unwrapped phase difference

$$\varphi(\tau) = \vartheta(t_2) - \vartheta(t_1) = \int_{t_1}^{t_2} \frac{\mathrm{d}\phi(t')}{\mathrm{d}t'}\,\mathrm{d}t', \tag{2}$$

where $\mathrm{d}\phi(t')/\mathrm{d}t'$ is the phase derivative at intervening times, $\vartheta(t)$ is an unwrapped phase value and $\tau = t_2 - t_1$. The statistical properties of the wrapped phase may differ considerably from those of the unwrapped phase. For example, the wrapped phase of a circular complex Gaussian field constitutes a stationary process, whereas the unwrapped phase is a non-stationary process with divergent variance. However, unwrapped phase fluctuations can be characterized by their mean square phase difference, or structure function, which can be obtained from (2) provided that the scattered field is once differentiable [10, 11]. In this paper we therefore focus our attention on the mean square properties of wrapped and unwrapped phase differences returned from the two-dimensional random walk scattering model (1). A correlation function follows immediately in the case of wrapped phase, which remains stationary, though not necessarily unwrapped phase since this quantity can become non-stationary.

2. Wrapped phase differences

The simplest field generated by the random walk scattering model (1) is that returned from a single scatterer. Assuming that the scattering amplitude a is fixed, the detected fields wrapped phase ϕ will be equivalent to the single scatterer's wrapped phase shift. In this paper we shall adopt a jointly-Gaussian model for the scatterer phase shifts θ [4]

$$p(\theta(t_2), \theta(t_1)) = \frac{1}{2\pi\langle\theta^2\rangle\sqrt{1-\rho_\theta^2(\tau)}}\exp\left\{-\frac{[\theta^2(t_2) + \theta^2(t_1) - 2\theta(t_2)\theta(t_1)\rho_\theta(\tau)]}{2\langle\theta^2\rangle(1-\rho_\theta^2(\tau))}\right\}, \tag{3}$$

where $\rho_\theta(\tau) = \langle\theta(t_2)\theta(t_1)\rangle/\langle\theta^2\rangle$ is the scatterer phase shift correlation function. The detected field's wrapped phase difference density in the single scatterer case can be evaluated from (3)

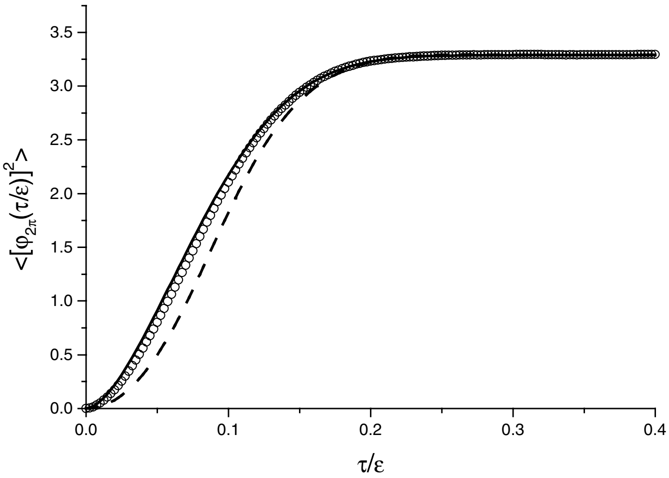

Figure 1. Mean square wrapped phase difference from 1 scatterer (dashed line), 3 scatterers (circles representing simulation results) and the large scatterer number Gaussian limit (solid line).

by changing to sum and differences coordinates and reducing the difference density modulo-2π [12], resulting in

$$p(\varphi_{2\pi}(\tau))_{N=1} = \frac{1}{2\pi}\left\{1 + 2\sum\nolimits_{k=1}^{\infty} \exp[-k^2\langle\theta^2\rangle(1-\rho_\theta(\tau))]\cos[k\varphi_{2\pi}(\tau)]\right\}. \tag{4}$$

Given an arbitrary wrapped phase difference density $p(\varphi_{2\pi}(\tau))$, its associated mean square wrapped phase difference follows by evaluating

$$\langle[\varphi_{2\pi}(\tau)]^2\rangle = \int_{-\pi}^{\pi} \varphi_{2\pi}^2(\tau)p(\varphi_{2\pi}(\tau))d[\varphi_{2\pi}(\tau)], \tag{5}$$

which returns the mean square wrapped phase difference from a single scatterer

$$\langle[\varphi_{2\pi}(\tau)]^2\rangle_{N=1} = \frac{\pi^2}{3} + 4\sum\nolimits_{k=1}^{\infty} \frac{(-1)^k}{k^2}\exp[-k^2\langle\theta^2\rangle(1-\rho_\theta(\tau))]. \tag{6}$$

While this result applies for arbitrary scatterer phase shift variance $\langle\theta^2\rangle$, in this paper we are most interested in scattering systems that give rise to unbiased two-dimensional random walks. This condition is usually taken to be met when $\langle\theta^2\rangle > 10$, corresponding to scattering centres whose motions exceed the radiation wavelength. The above result (6) is plotted in figure 1 and can be seen to approximate $\pi^2/3$ at large separation times, corresponding to wrapped phase differences that are uniformly distributed on $[-\pi, \pi)$. For small separation times $\rho_\theta(\tau)$ can be replaced by its Maclaurin series expansion $\rho_\theta(\tau) \approx 1 - \tau^2|\ddot{\rho}_\theta(0)|/2 + \cdots$, then (6) approximates $\langle\theta^2\rangle|\ddot{\rho}_\theta(0)|\tau^2$.

Wrapped phase differences can also be evaluated from the field scattered by two scatterers with phase shifts θ_1 and θ_2. It was previously shown that when the scattering amplitudes are fixed and equal the detected fields phase derivative is $\dot{\phi} = (\dot{\theta}_1 + \dot{\theta}_2)/2$ [12], whose variance is half that of a single scatterer. The detected field's unwrapped phase follows as $\vartheta = (\theta_1 + \theta_2)/2$, where we have ignored constants of integration since they have no effect when evaluating phase differences. A double scatterer wrapped phase difference result can then be developed in the same manner as the above single scatterer result. However, as discussed in [12] the case of two scatterers with equal scattering amplitudes a is somewhat idealistic. In the following section we shall consider the case when the two scatterers have unequal scattering amplitudes.

Analytical difficulties arise when attempting to evaluate wrapped phase differences from fields scattered by three or more scatterers. However, results can be obtained for a large number, i.e. $N \to \infty$, of statistically independent and identical scatterers, when the scattered field is known to constitute a circular complex Gaussian process. It can be shown that the mean square wrapped phase difference is then given by [9]

$$\langle[\varphi_{2\pi}(\tau)]^2\rangle_{N\to\infty} = \frac{\pi^2}{3} - \pi \sin^{-1}[g(\tau)] + (\sin^{-1}[g(\tau)])^2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{g^{2k}(\tau)}{k^2}, \tag{7}$$

where $g(\tau) = \langle E(t_2)E^*(t_1)\rangle/\langle|E|^2\rangle = \exp(-\langle\theta^2\rangle[1-\rho_\theta(\tau)])$ is the appropriate field correlation function. This result saturates to $\pi^2/3$ over large separation times, and can be approximated by $-|\ddot{g}(0)|\tau^2\ln[\tau]$ in the small separation time limit. As shown in figure 1 this rises more sharply than the corresponding result for a single scatterer (6), a consequence of the large phase fluctuations that manifest in complex Gaussian fields.

Results for three and more scatterers can be obtained using the numerical simulation techniques described in [11, 12]. All graphical results presented in this paper were generated using a Gaussian phase shift correlation function $\rho_\theta(\tau) = \exp(-\tau^2/\xi^2)$, phase shift variance $\langle\theta^2\rangle = 100$ and correlation length $\xi = 200$. As shown in figure 1, the simulated result for three equal scatterers deviates slightly from that of a circular complex Gaussian field at small separation times. However, for four or more scatterers simulation results are found to be essentially identical those of a circular complex Gaussian field, indicating that the wrapped phase difference correlation converges rapidly with increasing scatterer number to that of Gaussian speckle.

3. Unwrapped phase differences

When only one scatterer is present the field's unwrapped phase is equivalent to the scatterer phase shift, and so the mean square unwrapped phase difference is $2\langle\theta^2\rangle(1-\rho_\theta(\tau))$. In the small separation time limit this unwrapped result is identical to (6), a consequence of the fact that the modulus of the unwrapped phase difference remains less than π, and so the wrapped phase never actually wraps.

Given that the unwrapped phase from two identical scatterers is half the sum of their individual phase shifts, the corresponding mean square unwrapped phase difference is half the single scatterer result. However, the case of two scatterers with exactly equal scattering amplitudes is a special one. Interference between two such scatterers can only produce field zeros and not fields that pass through and encircle the origin in the complex plane. In practice, however, any difference in the scattering amplitudes will cause the resultant field to encircle the origin. Following from [12], for two scatterers of scattering amplitude unity and $r > 1$ the detected field's phase derivative can be shown to be

$$\dot{\phi} = \dot{\theta}_2 + (\dot{\theta}_1 - \dot{\theta}_2)\frac{1 + r\cos[\theta_2-\theta_1]}{1 + r^2 + 2r\cos[\theta_2-\theta_1]} = \dot{\theta}_2 - \frac{\mathrm{d}}{\mathrm{d}t}\left\{\tan^{-1}\left[\frac{\sin[\theta_2-\theta_1]}{r+\cos[\theta_2-\theta_1]}\right]\right\}, \tag{8}$$

whose variance is $\langle\dot{\phi}^2\rangle = \langle\dot{\theta}^2\rangle r^2/(r^2-1)$ when $[\theta_2-\theta_1]$ reduced modulo-2π is uniformly distributed. The corresponding unwrapped phase follows as

$$\vartheta = \theta_2 - \tan^{-1}\left[\frac{\sin[\theta_2-\theta_1]}{r+\cos[\theta_2-\theta_1]}\right], \tag{9}$$

where we have again ignored constants of integration. The second term on the right-hand side corresponds to the phase of a complex quantity with components $X = (r + \cos[\theta_2-\theta_1])$ and $Y = \sin[\theta_2-\theta_1]$, which can be visualized as a unit phasor rotating around the point $\{x = r > 1, y = 0\}$. Recognizing that the phasor tip rotates only in the positive half-plane,

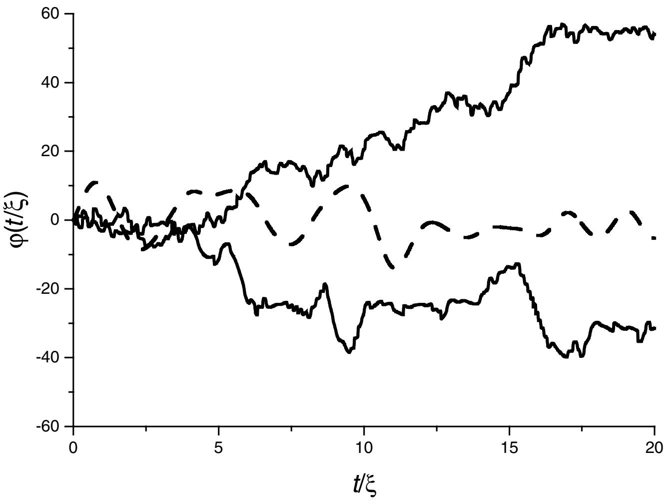

Figure 2. Realizations of unwrapped phase from 2 (dashed line), 3 (lower solid line) and 10 (upper solid line) scatterers.

its phase contribution to (9) cannot exceed $\pm\pi/2$. For phase shift variances $\langle\theta^2\rangle > 10$ the scatterer phase shift θ_2 will therefore dominate the detected field's unwrapped phase ϑ, and so we conclude that the mean square unwrapped phase difference from two unequal scatterers will approximate that of a single scatterer.

Analytical difficulties also arise when evaluating unwrapped phase differences from three or more scatterers. Results can, however, be obtained in the large scatterer Gaussian limit using the result [10, 11]

$$\langle[\varphi(\tau)]^2\rangle = 2\int_0^\tau (\tau - t)\rho_{\dot{\phi}}(t)\,\mathrm{d}t, \tag{10}$$

where $\rho_{\dot{\phi}}(t) = \langle(\mathrm{d}\phi(t')/\mathrm{d}t')(\mathrm{d}\phi(t'')/\mathrm{d}t'')\rangle$ is the phase derivative correlation function, and Rice's result for a circular complex Gaussian process [13]

$$\rho_{\dot{\phi}}(t) = \frac{1}{2}\frac{g(t)\ddot{g}(t) - \dot{g}(t)^2}{g^2(t)}\ln[1 - g^2(t)]. \tag{11}$$

Inserting the appropriate field correlation function obtains

$$\langle[\varphi(\tau)]^2\rangle_{N\to\infty} = \langle\theta^2\rangle\int_0^\tau (\tau - t)\ddot{\rho}_\theta(t)\ln[1 - \exp(-2\langle\theta^2\rangle[1 - \rho_\theta(t)])]\,\mathrm{d}t, \tag{12}$$

which for large separation times and $\langle\theta^2\rangle > 10$ can be shown to approximate

$$\langle[\varphi(\tau)]^2\rangle_{N\to\infty} = \frac{\sqrt{\pi\langle\theta^2\rangle|\ddot{\rho}_\theta(0)|}}{2}\varsigma(3/2)\tau - \frac{\pi^2}{12} \qquad \tau\to\infty, \tag{13}$$

where $\varsigma(n)$ is the Riemann-zeta function [14]. While the unwrapped phase of one and two scatterers remain stationary, this result indicates that the unwrapped phase in the large scatterer number Gaussian limit constitutes a non-stationary process. The small time limit to (12) can be shown to agree with the corresponding wrapped phase result (7).

The nature of the non-stationary transition that occurs between 2 and a large number of scatterers can be investigated using the numerical simulation technique. Figure 2 plots individual unwrapped phase realizations from 2, 3 and 10 scatterers, which shows that the

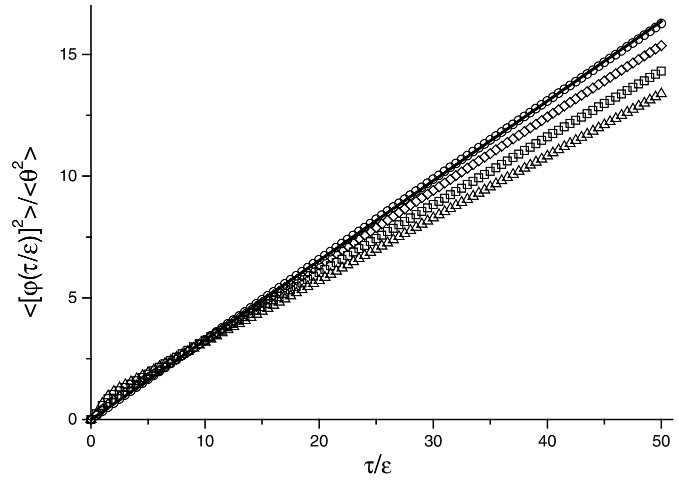

Figure 3. Simulation results for the mean square wrapped phase difference from 3 (triangles), 4 (squares), 10 (diamonds) and 50 (circles) scatterers. The solid line is the analytical prediction of a Gaussian speckle field.

unwrapped phase from 3 and 10 scatterers undergoes a large number of abrupt phase changes. Figure 3 plots mean square unwrapped phase differences from 3, 4, 10 and 50 scatterers, demonstrating that the unwrapped phase from three and more scatterers constitute non-stationary processes. It is also apparent that, as the number of scatterers is increased, the mean square unwrapped phase differences converge much more slowly to the Gaussian limit than the corresponding wrapped results shown in figure 1.

4. Discussion and conclusions

In this paper, we investigated wrapped and unwrapped phase differences generated by a two-dimensional random walk arising from wave scattering by N discrete scatterers. The unwrapped phase was found to be more sensitive to scatterer number than wrapped phase. Indeed, a major change is found to occur when N increases from two to three: the unwrapped phase transitions from a stationary process for one and two scatterers, to a non-stationary process for three or more scatterers. The transition to non-stationary behaviour that takes place between two and three scatterers can be understood by considering the case of two unequal scatterers. It was shown that the unwrapped phase of the field scattered from two unequal scatterers (9) is equal to one of their phase shifts θ_2 plus the phase contribution from a unit phasor rotating around the point $\{x = r > 1, y = 0\}$, which is bounded on $[-\pi/2, \pi/2)$. Thus if θ_2 is stationary, so is the unwrapped phase. From this context one can see that if the length of this second phasor is allowed to vary randomly beyond unity, so that it can rotate into the negative half-plane and around the origin, the unwrapped phase will undergo large random phase jumps in addition to the phase change from the scatterer phase shift θ_2. It is these additional random phase changes that result in the phase transitioning to a non-stationary process and becoming equivalent to a one-dimensional random walk with uncorrelated steps. As seen in figure 2, this is what happens in the field scattered from three scatterers, which is equivalent to one phasor of fixed length and a second phasor whose length varies randomly as the sum of two scattered contributions. The same argument applies to four or more scatterers.

While the results of this paper are directly applicable to scattering from discrete points, they also provide insights into more general scattering situations. In waves propagating through random media, or scattered from rough surfaces, the scattered field is often represented as the vector addition of a coherent component and an incoherent or scattered contribution. The generality of the above argument suggests that whenever the magnitude of the randomly varying scattered contribution has a finite probability of exceeding the coherent component, the resultant field's unwrapped phase will constitute a non-stationary process. Such conditions are met by the familiar Rice, homodyned-K and generalized-K models, commonly adopted scattering models that reduce to the circular complex Gaussian and K-processes in the appropriate limit [15].

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